Mimetic Discretizations of Diffusion Equation on Polygonal Meshes

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As the mathematical modeling of a fluid flow becomes more sophisticated, the need for discretization methods handling meshes with mixed types of elements has appeared. Practice and experience show that the most effective discretization methods mimic the underlying properties of original continuum differential operators. For the linear diffusion problem such methods mimic the Gauss divergence theorem needed for local mass conservation, the symmetry between the continuous gradient and divergence operators needed for proving symmetry and positivity of the resulting discrete operator, and the null spaces of the involved operators needed for stability of the discretization.

We have developed a new family of mimetic discretizations [1] for diffusion-type equations on general polygonal meshes:

$$\operatorname{div} \mathbf{u} = Q\,,$$

$$\mathbf{u} = -\mathrm{K} \operatorname{grad} p\,.$$

Here p and **u** denote the fluid pressure and velocity, respectively, K denotes a full tensor, and Q denotes a source function.

The novel discretization is locally conservative and exact for piecewise linear solutions (see Fig. 1). This is one of the major advances over the capabilities of the existing discretizations [2]. For sufficiently smooth solutions, our method exhibits a second-order convergence rate for the fluid pressure and a first-order convergence rate for the fluid velocity. It confirms the convergence rates observed in other types of lower order discretizations (finite elements and finite volumes) on nonsmooth triangular and quadrilateral meshes.

Another important feature of our method is the ability to treat meshes with degenerate and nonconvex polygons (see Fig. 2). Such meshes frequently occur in applications. For example, nonmatching meshes and locally refined meshes with hanging nodes are examples of conformal polygonal meshes. Recall that a hanging node occurs when two (or more) elements share an edge with one element. If we consider the hanging node as an additional vertex of this element, we get a conformal polygonal mesh with degenerate elements. As shown in Fig. 3, our method allows a very strong mesh refinement and elements with very small edges.

Nowadays, the use of polygonal meshes is limited by a small number of accurate discretization schemes. We mention here the finite volume scheme proposed by T. Palmer [3]. The scheme is exact for uniform flows but results in a nonsymmetric

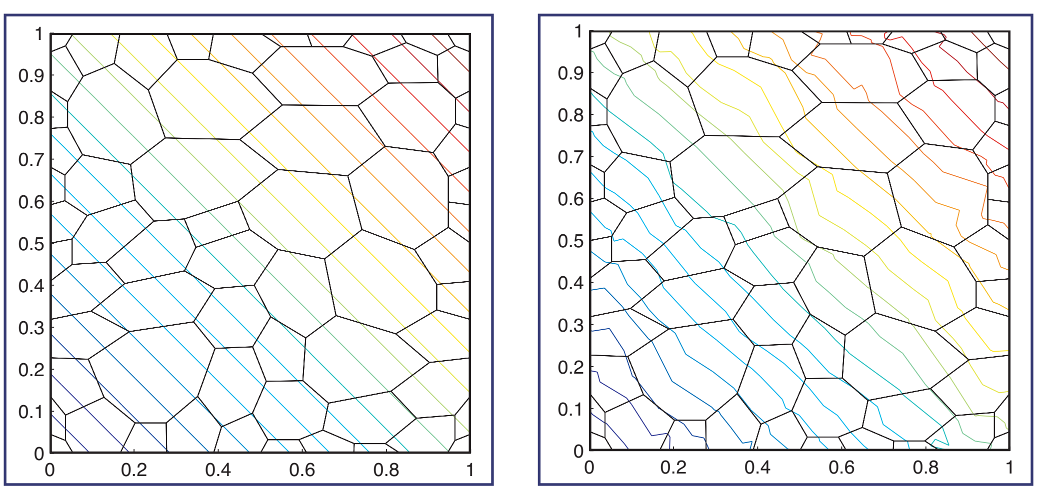

Figure 1— ***Isolines of a linear solution in the old (right) and new (left) methods.***

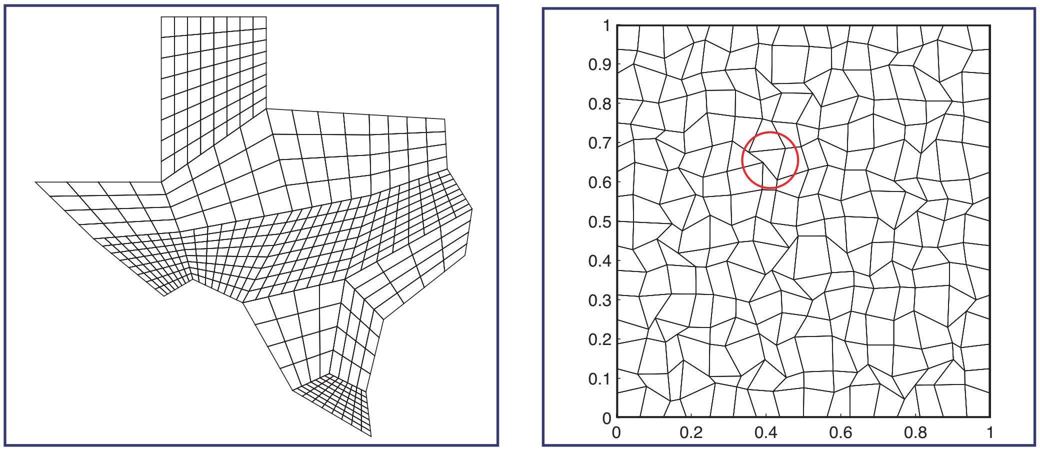

Figure 2—
Polygonal meshes with degenerate and nonconvex polygons. One of the nonconvex elements is marked by a circle.

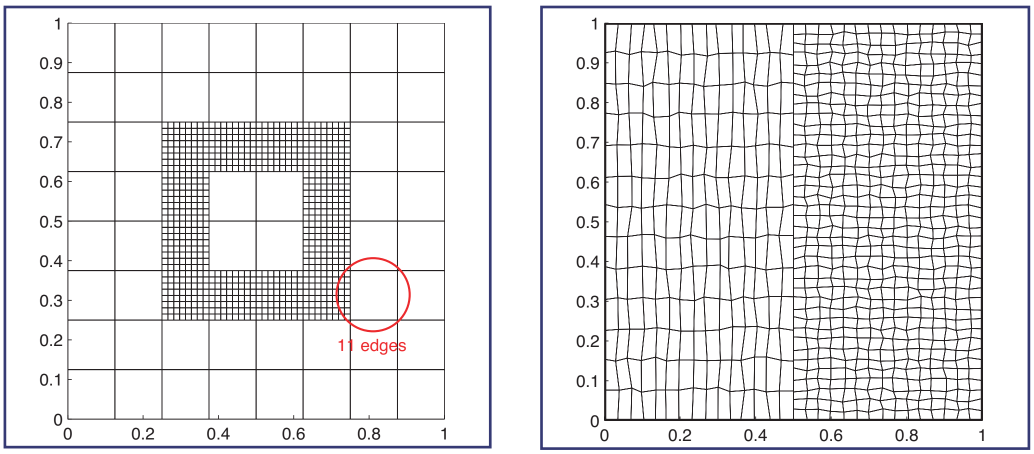

Figure 3—
Left picture shows a mesh with a strong local refinement. The element with 11 edges is marked by a circle. The right picture shows a non-matching mesh. Some of the polygons have very small edges.

coefficient matrix. Therefore, it requires the use of nontraditional iterative solvers. In contrast, our new discretization method results in an algebraic problem with a symmetric positive definite matrix. Therefore, the problem may be solved with the conjugate gradient method.

Our newly developed discretization methodology is based on the divide and conquer principle. First, we consider each mesh polygon as an independent domain and generate an independent discretization for this polygon. Second, the system of element-based discretizations is closed by imposing boundary conditions and continuity conditions for the fluid pressure and normal velocity component on polygon edges.

The new discretization methodology can be extended to unstructured polyhedral meshes and to other partial differential equations such as Maxwell's equations, Navier-Stokes equations, and equations of linear elasticity.

[1] Y. Kuznetsov, K. Lipnikov, M. Shashkov, "The Mimetic Finite Difference Method on Polygonal Meshes for Diffusion-Type Problems," Los Alamos National Laboratory report LA-UR-03-7608 (October 2003), accepted to *Comp. Geosciences*.
[2] J. Hyman et al., "Mimetic Finite Difference Methods for Diffusion Equations," *Comp. Geosciences* **6** (3–4) (2002), 333–352.
[3] T. Palmer, "Discretizing the Diffusion Equation on Unstructured Polygonal Meshes in Two Dimensions," *Ann. Nucl. Energy*, **28** (2001), 1851–1880.

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